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Inverse Mode Shape Problem For Bars and Beams with Flexible Supports

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ABSTRACT

Designing a structure for desired mode shapes, in addition to natural frequencies, is of interest in applications such as resonant-mode micro sensors and actuators, acoustics, manufacturing tooling, etc. This paper focuses on the inverse problem of determining the cross-section profile of a bar or a beam of a given homogeneous material so that it has a prescribed eigenmode shape. It is known that the range of mode shapes for a structure is restricted by its boundary conditions. After reviewing the reason underling this restriction, flexible supports are considered to expand the range of valid mode shapes. Both continuous and discretized models are considered, and closed-form analytical solutions (wherever possible) and approximate numerical solutions are discussed. In particular, cubic shapes are used to illustrate the analytical solution method for the case of bars.

INTRODUCTION

The natural frequencies and normal mode shapes of structural members play an important role in machines that operate at high speeds or for high precision. Some devices, especially at the micro scale, are intentionally operated at resonance frequencies to enhance their performance. Examples of such devices include microsensors and microactuators. For instance, in the micro rate gyroscope [1], the mode shape of a ring structure can be optimized to improve the performance. Designing the shape of the cantilever probes on the atomic force microscope to attain a desired modal deflection is another example [2]. Mode shapes of a vibrating structure are also important in the design of ultrasonic motors [3]. At the macro scale, accoustic noise reduction can be achieved by suitably designing the structural member that is interacting with the noise-propagating fluid medium [4]. In manufacturing tooling, it is preferred to have certain parts of the fixtures not move due to external vibrations. There too, designing the mode shapes so that those parts do not move significantly even though the structure itself might vibrate. This work is motivated by all of these aforementioned applications. The focus of this paper is on simple structures such as axially vibrating bars and transversely vibrating beams. These two simple cases are rich enough to illustrate the principal features of such a design task in the context of solving it an an inverse eigenmode problem for given material.

The “inverse frequency” problem has been studied extensively [5], including the topology optimization problem [6, 7]. In that problem, an elastic structure is to be sized and shaped to have a prescribed natural frequency. The “inverse mode shape” problem, on the other hand, entails the determination of the size and shape of the structure such that it will have prescribed mode shapes. Unlike the inverse frequency problem, the inverse mode shape problem has received much less attention.

Inverse mode shape shape problems can be classified into two distinct groups. In the first category, experimentally obtained eigendata (natural frequencies and mode shapes) of the structures is used for the characterization of their geometry *and* material density. Efforts in this direction are found in [8-16]. In the second category, which is the focus of this paper, the geometry of the structure is designed for prescribed mode shapes using a *given* material. Efforts in this direction are found in [2, 17]. It should be noted that an arbitrarily specified mode shape might not always be physically realizable with a given class of structures such as strings, cables, rods, frames consisting of straight and curved beams, membranes and plates, shells, and general 3-D structures. Consequently, in the second category of “the design for desired mode shape problems", the prescribed mode shape should be checked against a set of criteria that ensure physical realizability.

In the following sections, a brief description of the related work and our prior work are described. The motivation for flexible end consitions (i.e., spring supports) is then presented. The problem of the bars with flexible supports is presented in detail with analytical solutions as well as numerical solution for the discretized problem. The beam problem is presented in the section following that. Concluding remarks are in the last section.

RELATED WORK

Barcilon incorporated eigenvector data to reconstruct the physical parameters (spring constants and masses) in a discrete spring-mass model of a bar from three eigenvectors and their corresponding eigenvalues associated with three boundary conditions [18-20]. Ross presented a procedure for deriving both the mass and stiffness matrices from experimentally measured natural frequencies and a square modal matrix composed of measured mode vectors supplemented by arbitrary linearly independent vectors [8]. Gladwell derived the necessary and sufficient conditions applicable to the spectral data (eigenvalues and the corresponding eigenvectors) to permit the construction of a realizable beam [21]. Ram and Caldwell showed that the physical parameters of a free multi-connected spring-mass system could be determined from certain spectral sequences as well [12]. The inverse mode problem for the continuous model of an axially vibrating bar from two eigenvalues, the corresponding eigenvectors and the total mass of the system was solved by Ram [14]. Ram and Gladwell proposed a way to reconstruct a finite element model of a vibrating bar from a single eigenvalue and two eigenvectors based on the fact that both the mass and stiffness matrices of the finite element model are tri-diagonal [16]. Ram also showed a method of reconstructing a finite difference model of a vibrating beam from three eigenvectors, one eigenvalue and the total mass of the beam [15].

The motivation of Ram and Gladwell's work, and other references above was to recover the density and shape information from experimental eigendata of the structure. This differs from the focus of this paper where the density is constant throughout the bar/beam. This has two consequences: (a) the physical parameters pertaining to springs and masses cannot be independent as they both depend on the area of cross-section, (b) there will be further restrictions on the range of valid mode shapes. It was discussed in [17] where fixed-free and fixed-fixed bars, and fixed-free beam were considered. The next section describes the problem determining the cross-section shape of the bar/beam for a given homogeneous material (i.e., the density and Young's modulus are constant throughout).

MOTIVATION AND PROBLEM DEFINITION

Consider an axially deforming bar of cross-section profile, $A(x)$, made of homogeneous material of Young's modulus, E, and mass density, ρ. Figure 1 shows three types of boundary conditions that a bar can have, viz. fixed, free, and flexible supports. The support boundary conditions at the two ends naturally impose limits on the possible eigenmode shapes for the bar. By denoting the axial deformation as $A(x)$, the boundary conditions can be written as follows.

Fixed: $$u(x^*) = 0 \tag{1a}$$

Free: $$u'(x^*) = \left.\frac{du}{dx}\right|_{x=x^*} = 0 \tag{1b}$$

Flexible: $$EA(x^*)u'(x^*) = k_s u(x^*) \tag{1c}$$

where x^* is either 0 or L, with L denoting the length of the bar, and k_s the spring constant of the flexible support. The governing equation for the small-amplitude free vibrations of a bar is given by

$$(EAu')' + \lambda\rho A u = 0 \tag{2}$$

where λ is the square of the natural frequency, ω, of the natural vibration, $u(x)$ is the mode shape, and $(\cdot)'$ indicates derivative w.r.t. x.

When a mode shape is prescribed, the area profile can be determined by re-writing Eq. (2) with $A(x)$ as the unknown.

$$Eu'A' + (Eu'' + \lambda\rho u)A = 0 \tag{3}$$

The solution of the above equation is given by

$$A = Ce^{-\int \psi\, dx} \tag{4a}$$

where

$$\psi = \frac{Eu'' + \lambda\rho u}{Eu'} \tag{4b}$$

and C is determined with the constraint on the total mass of the bar. As discussed in [17], when $u(x)$ is given, λ is automatically determined so that the numerator in Eq. (4b) is zero or the denominator is cancelled of as a factor of the numerator. This is necessary because both the fixed-free and fixed-fixed boundary conditions will make u' zero for some $x \in [0, L]$. When that happens, $\psi \to \infty$, which makes the resulting area profile not finite and hence physically realistic.

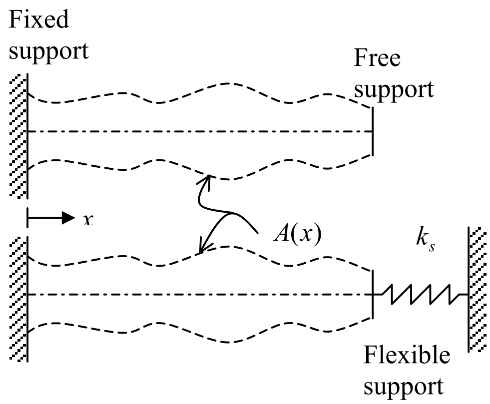


Fig. 1. Boundary conditions of an axially deforming bar.

It should also be noted that for some prescribed shapes satisfying the fixed-free or fixed-fixed bundary conditions, it may not be possible to find a suitable λ, which makes them invalid mode shapes. The same is true for transfersely vibrating beams. In fact, the criteria for valid mode shapes are even more stringent and less understood for beams [17].

As an example, consider a cubic polynomial as a mode shape for a fixed-free bar. In order to satisfy the boundary conditions, it should have the following form for a nomalized bar of $L = 1$.

$$u = c_3 x^3 + c_2 x^2 - (3c_3 + 2c_2)x \qquad (5)$$

In order to obtain all possible cubics of the above form in $x \in [0,1]$, one of the two conditions can be imposed: (i) $u'(s) = 0$ or (ii) $u(s) = f$ at some $s \in [0,1]$ where $0 \le f \le 1$. In either case, by normalizing u so that $\max(u)$ is unity, the curves shown in Fig. 2 are obtained. Out of all these, only two, shown with thick curves in Fig. 2, are valid mode shapes for a fixed-free bar. These are determined by making the denominator of Eq. (4b) a factor of the numerator. They are given by the solution $s \in (0,1)$ of the following equation:

$$\begin{aligned} &2s^6 - 18s^5 + 51s^4 - (54 - 10f)s^3 + \\ &(18 + 18f) - 6sf - f^2 = 0 \end{aligned} \qquad (6)$$

This leads to only two thick curves shown in Fig. 2. Their corresponding area profiles are shown in Fig. 3, the top area for the valid mode shape without the slope being zero except at $x = 1$ (labeled 1) and the bottom area for the valid shape with $u' = 0$ at $x = 2 - \sqrt{3}$ in addition to $x = 1$ (labeled 2). The frequencies of these two valid mode shapes are equal and are given by

$$\omega_1 = \omega_2 = \sqrt{\lambda} = \sqrt{\frac{6E(2 + \sqrt{3})}{\rho}} \qquad (7)$$

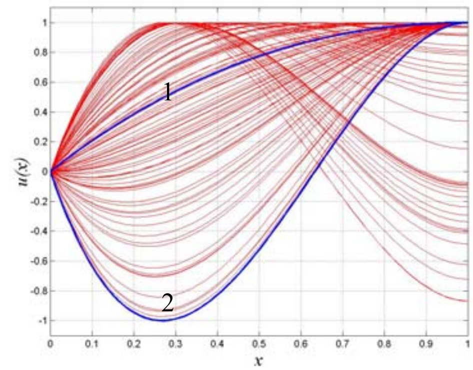


Fig. 2 Possible cubic shapes for a fixed-free bar. The two valid mode shapes are shown as thick curves.

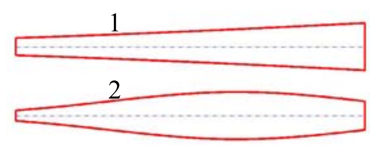


Fig. 3 Area profiles of two valid cubic mode shapes shown in Fig. 2.

Similar analysis for the fixed-fixed case reveals that only three are valid mode shapes (shown as thick curves) out of the many possible shapes. In fact, two out of these are the same due to the symmetry of the fixed-fixed boundary condition.

As can be seen above, the valid mode shapes are rather limited. Therefore, flexible supports are considered next to expand the range of valid mode shapes.

FLEXIBLE SUPPORT FOR A BAR

Consider the bar with fixed and flexible support conditions shown in Fig. 1. To illustrate how the range of valid shapes can be expanded, consider the case of cubic shapes once again. Then, the boundary conditions, Eqs. (1a) and (1c), give rise to the following cubic:

$$u = c_3 x^3 + c_2 x^2 + c_1 x \tag{8}$$

where

$$L^2 \frac{c_3}{c_1}\left(\frac{1-3f}{1-f}\right) + L\frac{c_2}{c_1}\left(\frac{1-2f}{1-f}\right) + 1 = 0 \tag{9}$$

with

$$f = \frac{A_L E}{k_s L} \tag{10}$$

A_L denoting the area of cross-section at $x = L$. It should be noted that c_1 is immaterial as it simply helps in normalizing the mode shape to have its maximum value equal to unity. Several values for c_2 and c_3 give a vast range of shapes but there will be a further restriction on them in order to make ψ in Eq. (4b) finite.

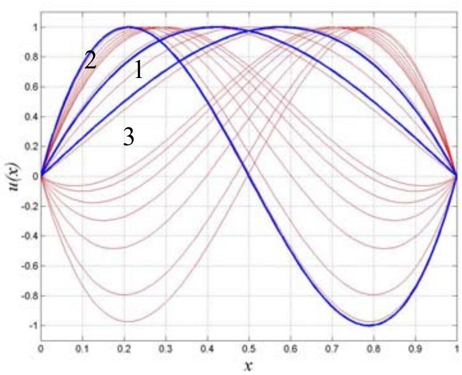


Fig. 4. Possible cubic shapes for a fixed-fixed bar. The three valid mode shapes are shown as thick curves.

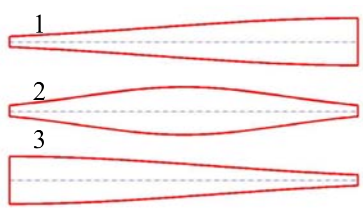


Fig. 5. Area profiles of two valid cubic mode shapes shown in Fig. 4

Noting that the numerator of ψ is a cubic while the denominator is a quadratic, if the numerator cancels off as a factor of the numerator, ψ will simplify to

$$\psi = \frac{Eu'' + \lambda \rho u}{Eu'} = p + qx \tag{11}$$

Then, the coefficients of x^3, x^2 and x, and the constant term in the following equation are zero.

$$Eu'' + \lambda \rho u - (p + qx)Eu' = 0 \tag{12}$$

Three out of the resulting four equations can be used to solve for p, q and λ.

$$p = 2\frac{c_2}{c_1} \tag{13a}$$

$$q = 6\frac{c_3}{c_1} \tag{13b}$$

$$\lambda = \frac{3E}{\rho} q = \frac{18E}{\rho}\frac{c_3}{c_1} \tag{13c}$$

The fourth equation, corresponding to the coefficient of x, gives

$$27c_3^2 + 18c_3c_2 + 2c_2^2 = 0 \tag{13d}$$

Substitution of Eqs. (13a) and (13b) into (13d), when $c_2 \neq 0$, yields

$$p = \frac{1}{q}\left(-1 \pm \sqrt{\frac{1}{3}}\right) \tag{14}$$

As pointed out earlier, c_2 and c_3 cannot be arbitraily chosen as they are constrained by Eq. (13d). Consequently, only one of them can be varied freely. It is an improvement over the fixed-free and fixed-fixed cases because many mode cubics are now valid mode shapes, as explained next.

Let c_3, or equivalently q, be the independent variable to obtain a wide range of valid mode shapes. It should be noted that choosing c_3 (or q) automatically determines λ, and hence the natural frequency, ω. Alternatively, choosing ω fixes the possible cubic mode shape. Although it may seem restrictive, it is worth noting that now there exists a cubic mode shape for any frequency for a chosen material. The following steps summarize the procedure of obtaining all such cubic mode shapes and the corresponding physical parameters including the area profile and the spring constant of the flexible support using Eqs. (4a), (9), and (10).

Step 1. Choose either c_3 (mode shape) or λ (natural frequency). If c_3 is chosen, two possible values for c_2 are computed using Eq. (13d), and c_1 is determined such that the mode shape is normalized so that the maximum value is unity.

Step 2. Determine q using Eq. (13c).

Step 3. Determine p using Eq. (14).

Step 4. Area profile is given by Eq. (4a) so that the area at the flexible support end is A_L. That is,

$$A = \frac{A_L}{e^{-(pL+0.5qL^2)}} e^{-(px+0.5qx^2)} \tag{15}$$

Step 5. Using Eqs. (9) and (10), obtain p:

$$f = \frac{6+3Lp+L^2q}{-6Lp+3L^2q} \tag{16}$$

Step 6. Determine the spring constant of the flexible support using

$$k_s = \frac{A_L E}{fL} \tag{17}$$

Using the above procedure, a number of shapes that were previously invalid can now be obtained as valid mode shapes. Figure 6a shows several such shapes. For the thick curve in Fig. 6a, the area profile is shown in Fig. 6b.

(a)

(b)

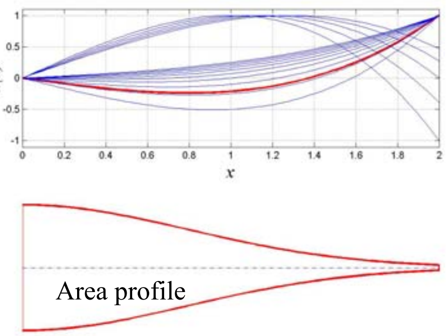


Fig. 6 (a) A variety of possible cubic mode shapes for a bar with fixed and flexible supports (b) the area profile for the shape shown as a thick curve.

(a)

(b)

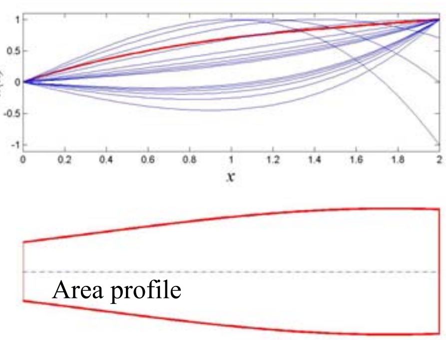


Fig. 7 (a) A variety of possible cubic mode shapes without the x^2 term for a bar with fixed and flexible supports (b) the area profile for the shape shown as a thick curve.

The special case of $c_2 \neq 0$ has slightly different relationships for p and q,

$$p = 0,\ q = -3\frac{c_3}{c_1} \tag{18}$$

but the rest of the procedure is the same. Such cubic mode shapes and one example area profile are shown in Figs. 7a and 7b.

This type of analysis can be done for any assumed shape (polynomial of certain degree or other types of functions). But it can become cumbersome, as the functions get complicated. So, a numerical solution procedure is presented below for any type of prescribed mode shape.

NUMERICAL SOLUTION FOR A BAR

Using the finite element method, the bar fixed at the left end and with a flexible support at the right end can be discretized into N elements, each with an area $A_i\,(i = 1,2,\cdots N)$ and length $l = L/N$. Then, the stiffness and inertia matrices, $\mathbf{K}$ and $\mathbf{M}$ respectively, are given by [22]

$$\mathbf{K} = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdots & 0 \\ 0 & -k_3 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & k_{N-1}+k_N & -k_N \\ 0 & 0 & \cdots & -k_N & k_N+k_s \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 2m_1+2m_2 & m_2 & 0 & \cdots & 0 \\ m_2 & 2m_2+2m_3 & m_3 & \cdots & 0 \\ 0 & m_3 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & 2m_{N-1}+2m_N & m_N \\ 0 & 0 & \cdots & m_N & 2m_N+k_s \end{bmatrix}$$

where

$$k_i = \frac{A_i E}{l_i} \tag{19a}$$

$$m_i = \frac{\rho A_i l_i}{6} \tag{19b}$$

The discrete counterpart of the continuous eigenvalue problem (Eq. (2)) is obtained as

$$\mathbf{KU} = \lambda \mathbf{MU} \tag{20}$$

where $\mathbf{U}$ is the $N \times 1$ column vector consisting of the axial displacements of each of the right side nodes of the N finite elements.

Just as Eq. (2) was re-arranged as Eq. (3) to make the area the unknown, Eq. (20) can be re-arranged to make the column vector $\mathbf{A}$ as the unknown where i th entry of $\mathbf{A}$ is A_i.

$$\mathbf{PA} = \mathbf{Q} \tag{21}$$

where

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & 0 & \cdots & 0 \\ 0 & p_{22} & p_{23} & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & p_{(N-1)(N-1)} & p_N \\ 0 & 0 & \cdots & 0 & p_{NN} \end{bmatrix} \tag{22a}$$

$$\mathbf{Q}^T = \{0 \quad 0 \quad \cdots \quad 0 \quad -k_s u_N\} \tag{22b}$$

$$p_{11} = \frac{Eu_1}{l} - \frac{\lambda\rho l u_1}{3}$$
$$p_{12} = \frac{E(u_1 - u_2)}{l} - \frac{\lambda\rho l(2u_1 + u_2)}{6} \tag{22c}$$

for $j = 2,3,\cdots(N-1)$,

$$p_{jj} = \frac{E(u_j - u_{j-1})}{l} - \frac{\lambda\rho l(2u_j + u_{j-1})}{6}$$
$$p_{12} = \frac{E(u_j - u_{j+1})}{l} - \frac{\lambda\rho l(2u_j + u_{j+1})}{6} \tag{22d}$$

$$p_{NN} = \frac{E(u_N - u_{N-1})}{l} - \frac{\lambda\rho l(2u_N + u_{N-1})}{6} \tag{22e}$$

Now, the structure of the linear equations in Eq. (21) readily allows computation of the areas as follows.

$$A_N = \frac{-k_s u_N}{p_{NN}} \tag{23a}$$

$$A_j = \frac{-p_{j(j+1)}}{p_{jj}} A_{j+1} \quad \text{for} \quad j = (N-1),\cdots,1 \tag{23b}$$

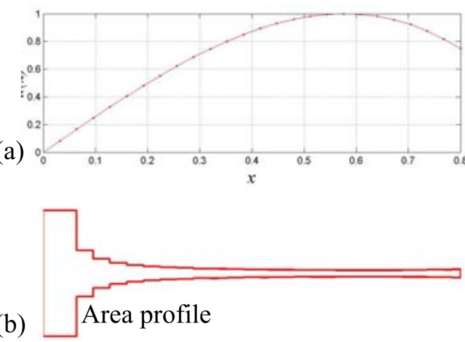
Thus, the areas of all the elements can be determined. An example is shown in Fig. 8 for the data given in Table 1. The mode shape and the area profile are shown in Figs. (8a) and (8b) respectively.

Table 1. Data for Fig. 8

ρ	$= 7800\ kg/m^3$
E	$= 210\ GPa$
L	$= 0.8\ m$
ω	$= 100\ kHz$
N	$= 25$
k_s	$= \frac{\pi(0.01)^2 E}{0.01} = 6.6\text{E}9\ N/m$

It is important to note that the success of the above procedure depends on all areas turning out to be positive for a given discretized mode shape $\mathbf{U}$, material properties (E, ρ), chosen natural frequency ($\lambda = \omega^2$), and chosen spring constant of the flexible support (k_s). When anyone of the elements' areas of cross-section is not positive, it means that either the given mode shape is not valid or that the parameters chosen are inconsistent. Fortunately, the step-wise "back substitution" nature of the solution (Eqs. (23a) and (23b)) enables correcting both the parameters and, if necessary, the mode shape as well.

(a)



(b)

Fig. 8 (a) A prescribed mode shape for a bar with fixed and flexible supports (b) the area profile for the shape obtained using the numerical method.

FLEXIBLE SUPPORTS FOR BEAMS

The equation governing the small-amplitude transverse, free vibrations of a beam is given by

$$(EIw'')'' - \lambda\rho A w = 0 \tag{24}$$

where $w(x)$ is the transverse deformation of the beam, $I(x)$ is the moment of inertia, and the rest of the symbols have the same meaning as in the case of bars. Furthermore, for most cross-sections $I(x)$ can be expressed as a function of $A(x)$.

$$I(x) = \alpha A^n \tag{25}$$

Then, Eq. (24) can be re-arranged to have A as the unknown function for the case of $n = 1$ (e.g., rectangular cross-section with constant depth but variable width) as

$$E\alpha w'' A'' + 2E\alpha w''' A' + (E\alpha w^{iv} - \lambda\rho w)A = 0 \tag{26}$$

Unlike Eq. (3), the above second order equation cannot be analytically solved in closed form. So, checking if a given mode shape is valid is not as straightforward. Using some properties of mode shapes of beams, four criteria were proposed for cantilever (fixed-free) beams in [17]. It was also shown with a sixth degree polynomial that valid mode shapes are rather limited. Flexible supports, as shown in Fig. 9, help increase the range of valid mode shapes just as in the case of bars. Although analytical solutions are difficult to come by, numerical solution is similar to that of the bars presented in the paper.

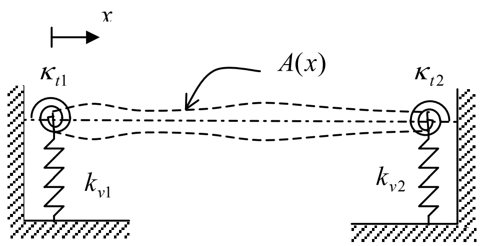


Fig. 9. Flexible supports for a beam

CLOSURE

The inverse problem of eigenmode shapes of bars and beams made of homogeneous material but variable cross-section is considered in this paper. Uniform density throughout the bar/beam differentiates this paper from the earlier work. Usual fixed or free boundary conditions severely limit the range of valid mode shapes. Hence, using flexible supports, it is shown for the case of bars that a much wider variety of mode shapes can be realized. It is worth noting that even then not all possible shapes are valid mode shapes. Furthermore, the frequency is usually fixed when a mode shape is prescribed. A numerical solution technique is presented. Analytical solutions for beams of different supports are yet to be explored.

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